Tutorial 1

MAU22E01

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Exercise 1

Find **v**+**u**, 3**u**, the length ||**u**||, ||**v**||, the dot product **u**·**v**, the angle between **u** and **v** and determine whether **u** and **v** are orthogonal (or for which values of parameters **u** and **v** are orthogonal, if any are present):

(i) **u**=(0,2), **v**=(2,1)

- **v**+**u**

$$\mathbf{v}+\mathbf{u} = (v_1 + u_1, v_2 + u_2)$$
$$\mathbf{v}+\mathbf{u} = (2 + 0, 1 + 2)$$
$$\mathbf{v}+\mathbf{u} = (2, 3)$$

- 3**u**

$$3\mathbf{u} = (3u_1, 3u_2)$$
$$3\mathbf{u} = (3 \times 0, 3 \times 2)$$
$$3\mathbf{u} = (0, 6)$$

- The length ||**u**||

$$||\mathbf{u}|| = \sqrt{u_1^2 + u_2^2}$$
$$||\mathbf{u}|| = \sqrt{(0)^2 + (2)^2}$$
$$||\mathbf{u}|| = \sqrt{4} = 2$$

- The length ||**v**||

$$||\mathbf{v}|| = \sqrt{v_1^2 + v_2^2}$$
$$||\mathbf{v}|| = \sqrt{(2)^2 + (1)^2}$$
$$||\mathbf{v}|| = \sqrt{4+1} = \sqrt{5}$$

- The dot product **u**·**v**

$$\mathbf{u} \cdot \mathbf{v} = u_1v_1 + u_2v_2$$
$$\mathbf{u} \cdot \mathbf{v} = (0)(2) + (2)(1)$$
$$\mathbf{u} \cdot \mathbf{v} = 2$$

- The angle between **u** and **v** and determine whether they are orthogonal

$$\mathbf{u}\cdot\mathbf{v} = ||\mathbf{u}||||\mathbf{v}||\cos\theta$$

$$\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{||\mathbf{u}||||\mathbf{v}||} = \frac{2}{2\sqrt{5}} = \frac{1}{\sqrt{5}}$$

$$\theta = \cos^{-1}\left(\frac{1}{\sqrt{5}}\right) \approx 63.43^{o}$$

u and **v** are not orthogonal as the angle bewteen them is not 90^{o}, or $\frac{\pi}{2}$. This is also clear from the non-zero value of **u**·**v** we obtained. Orthogonal vectors have **u**·**v**=0.

(ii) **u**=(-3,k,0,k), **v**=(0,3k,-1,3)

- **v+u**

$$\mathbf{v{+}u} = (v_1+u_1, v_2+u_2, v_3+u_3, v_4+u_4)$$

$$\mathbf{v{+}u} = (0-3, 3k+k, -1+0, 3+k)$$

$$\mathbf{v{+}u} = (-3, 4k, -1, 3+k)$$

- 3**u**

$$3\mathbf{u} = (3u_1, 3u_2, 3u_3, 3u_4)$$

$$3\mathbf{u} = (3\times -3, 3\times k, 3\times 0, 3\times k)$$

$$3\mathbf{u} = (-9, 3k, 0, 3k)$$

- The length $||\mathbf{u}||$

$$||\mathbf{u}|| = \sqrt{u_1^2+u_2^2+u_3^2+u_4^2}$$

$$||\mathbf{u}|| = \sqrt{(-3)^2+(k)^2+(0)^2+(k)^2}$$

$$||\mathbf{u}|| = \sqrt{9+k^2+k^2} = \sqrt{9+2k^2}$$

- The length $||\mathbf{v}||$

$$||\mathbf{v}|| = \sqrt{v_1^2+v_2^2+v_3^2+v_4^2}$$

$$||\mathbf{v}|| = \sqrt{(0)^2+(3k)^2+(-1)^2+(3)^2}$$

$$||\mathbf{v}|| = \sqrt{9k^2+1+9} = \sqrt{9k^2+10}$$

- The dot product **u**·**v**

$$\mathbf{u}\cdot\mathbf{v} = u_1v_1+u_2v_2+u_3v_3+u_4v_4$$

$$\mathbf{u}\cdot\mathbf{v} = (-3)(0)+(k)(3k)+(0)(-1)+(k)(3)$$

$$\mathbf{u}\cdot\mathbf{v} = 3k^2+3k$$

$$\mathbf{u}\cdot\mathbf{v} = 3k(k+1)$$

- The angle between **u** and **v** and determine whether they are orthogonal

$$\mathbf{u} \cdot \mathbf{v} = ||\mathbf{u}||||\mathbf{v}|| \cos\theta$$

$$\cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{||\mathbf{u}||||\mathbf{v}||} = \frac{3k(k+1)}{\sqrt{9+2k^2}\sqrt{9k^2+10}}$$

$$\theta = \cos^{-1}\left(\frac{3k(k+1)}{\sqrt{9+2k^2}\sqrt{9k^2+10}}\right)$$

u and **v** are orthogonal if **u**·**v**=0.

$$\mathbf{u} \cdot \mathbf{v} = 3k(k+1) = 0$$

So **u** and **v** are orthogonal if $k = 0$ or -1.

(ii) **u**=(1,0,0,-1,0,1), **v**=(0,2,0,0,-k,k)

- **v+u**

$$\mathbf{v}+\mathbf{u} = (v_1 + u_1, v_2 + u_2, v_3 + u_3, v_4 + u_4, v_5 + u_5, v_6 + u_6)$$
$$\mathbf{v}+\mathbf{u} = (1+0, 0+2, 0+0, -1+0, 0-k, 1+k)$$
$$\mathbf{v}+\mathbf{u} = (1, 2, 0, -1, -k, 1+k)$$

- 3**u**

$$3\mathbf{u} = (3u_1, 3u_2, 3u_3, 3u_4, 3u_5, 3u_6)$$
$$3\mathbf{u} = (3\times 1, 3\times 0, 3\times 0, 3\times -1, 3\times 0, 3\times 1)$$
$$3\mathbf{u} = (3, 0, 0, -3, 0, 3)$$

- The length $||\mathbf{u}||$

$$||\mathbf{u}|| = \sqrt{u_1^2 + u_2^2 + u_3^2 + u_4^2 + u_5^2 + u_6^2}$$
$$||\mathbf{u}|| = \sqrt{(1)^2 + (0)^2 + (0)^2 + (-1)^2 + (0)^2 + (1)^2}$$
$$||\mathbf{u}|| = \sqrt{1+1+1} = \sqrt{3}$$

- The length $||\mathbf{v}||$

$$||\mathbf{v}|| = \sqrt{v_1^2 + v_2^2 + v_3^2 + v_4^2 + v_5^2 + v_6^2}$$
$$||\mathbf{v}|| = \sqrt{(0)^2 + (2)^2 + (0)^2 + (0)^2 + (-k)^2 + (k)^2}$$
$$||\mathbf{v}|| = \sqrt{4+k^2+k^2} = \sqrt{4+2k^2}$$

- The dot product **u**·**v**

$$\mathbf{u} \cdot \mathbf{v} = u_1v_1 + u_2v_2 + u_3v_3 + u_4v_4 + u_5v_5 + u_6v_6$$

$$\mathbf{u} \cdot \mathbf{v} = (1)(0) + (0)(0) + (0)(0) + (-1)(0) + (0)(-k) + (1)(k)$$
$$\mathbf{u} \cdot \mathbf{v} = k$$

- The angle between **u** and **v** and determine whether they are orthogonal

$$\mathbf{u} \cdot \mathbf{v} = ||\mathbf{u}||||\mathbf{v}|| \cos\theta$$

$$\cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{||\mathbf{u}||||\mathbf{v}||} = \frac{k}{\sqrt{3}\sqrt{4+2k^2}}$$

$$\theta = \cos^{-1}\left(\frac{k}{\sqrt{3}\sqrt{4+2k^2}}\right)$$

u and **v** are orthogonal if **u**·**v**=0.

$$\mathbf{u} \cdot \mathbf{v} = k$$

So **u** and **v** are orthogonal if $k = 0$.

Exercise 2

Write the system in matrix form:

(i)

$$\begin{cases} z - 2z - 2y &= 0 \\ y + x &= 3 \end{cases}$$

$$\begin{pmatrix} 0 & -2 & -1 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \end{pmatrix}$$

(ii)

$$\begin{cases} 2z - 4t + x - 4y &= -1 \\ 2y &= 0 \\ z - t &= -3 \end{cases}$$

$$\begin{pmatrix} 1 & -4 & 2 & -4 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ -3 \end{pmatrix}$$

Exercise 3

Find the (standard) matrix of the linear transformations T defined by the equations:
(i) $w_1 = -x_1$, $w_2 = x_2 - x_1 + x_3$

The linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$ can be written in matrix form as: $\vec{y} = A\vec{x}$ or $T(\vec{x}) = A\vec{x}$, where A is the standard matrix.

$$\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} -x_1 \\ -x_1 + x_2 + x_3 \end{pmatrix}$$

$$\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} -1x_1 & + & 0x_2 & + & 0x_3 \\ -1x_1 & + & 1x_2 & + & 1x_3 \end{pmatrix}$$

$$\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 \\ -1 & 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

(ii) $w_1 = x - y + z$, $w_2 = z + y$, $w_3 = -x$, $w_4 = x$

$$\begin{pmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{pmatrix} = \begin{pmatrix} x - y + z \\ z + y \\ -x \\ x \end{pmatrix}$$

$$\begin{pmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{pmatrix} = \begin{pmatrix} 1x & + & -1y & + & 1z \\ 0x & + & 1y & + & 1z \\ -1x & + & 0y & + & 0z \\ 1x & + & 0y & + & 0z \end{pmatrix}$$

$$\begin{pmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & 1 \\ -1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

(iii) $T(x_1, x_2, x_3, x_4) = (0, x_1, x_3 - x_2, x_1 - 2x_4 + x_3, 0)$

$$\begin{pmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \\ w_5 \end{pmatrix} = \begin{pmatrix} 0 \\ x_1 \\ x_3 - x_2 \\ x_1 - 2x_4 + x_3 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \\ w_5 \end{pmatrix} = \begin{pmatrix} 0x_1 & + & 0x_2 & + & 0x_3 & + & 0x_4 \\ 1x_1 & + & 0x_2 & + & 0x_3 & + & 0x_4 \\ 0x_1 & + & -1x_2 & + & 1x_3 & + & 0x_4 \\ 1x_1 & + & 0x_2 & + & 1x_3 & + & -2x_4 \\ 0x_1 & + & 0x_2 & + & 0x_3 & + & 0x_4 \end{pmatrix}$$

$$\begin{pmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \\ w_5 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 1 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$$

Exercise 4

Find $T(\mathbf{x}) = A\mathbf{x}$ for the matrix A and the vector $\mathbf{x}$ whenever the product makes sense (i.e. the dimensions of A and $\mathbf{x}$ fit together):

(i) $A = \begin{pmatrix} 0 & -1 & 1 & -1 \\ 1 & -1 & 5 & 0 \end{pmatrix}$, $\mathbf{x}=\begin{pmatrix} 12 \\ -2 \end{pmatrix}$

We must compute the dot products of the rows of the 1^{st} matrix by columns of the 2^{nd}. We cannot do this for these matrices as the dimension of $\mathbf{x}$ is not equal to the dimension of the rows of A.

(ii) $A = \begin{pmatrix} 0 & 1 & -1 \\ 1 & 0 & -2 \\ -1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$, $\mathbf{x}=\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$

$$T(\mathbf{x}) = \begin{pmatrix} 0 & 1 & -1 \\ 1 & 0 & -2 \\ -1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

This is a 4×3 matrix by a 3× 1 matrix. The resulting matrix is a 4× 1 matrix.

$$T(\mathbf{x}) = \begin{pmatrix} (0 \times 0) + (1 \times 1) + (-1 \times -1) \\ (1 \times 0) + (0 \times 1) + (-2 \times -1) \\ (-1 \times 0) + (1 \times 1) + (1 \times -1) \\ (0 \times 0) + (1 \times 1) + (1 \times -1) \end{pmatrix}$$

$$T(\mathbf{x}) = \begin{pmatrix} 2 \\ 2 \\ 0 \\ 0 \end{pmatrix}$$

(iii) $A = \begin{pmatrix} 1 & 2 \\ -1 & 1 \\ 0 & -1 \\ -1 & 0 \end{pmatrix}$, $\mathbf{x}=\begin{pmatrix} 2 \\ -1 \end{pmatrix}$

$$T(\mathbf{x}) = \begin{pmatrix} 1 & 2 \\ -1 & 1 \\ 0 & -1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

This is a 4×2 matrix by a 2× 1 matrix. The resulting matrix is a 4× 1 matrix.

$$T(\mathbf{x}) = \begin{pmatrix} (1 \times 2) + (2 \times -1) \\ (-1 \times 2) + (1 \times -1) \\ (0 \times 2) + (-1 \times -1) \\ (-1 \times 2) + (0 \times -1) \end{pmatrix}$$

$$T(\mathbf{x}) = \begin{pmatrix} 0 \\ -3 \\ 1 \\ -2 \end{pmatrix}$$